Local Scaling in Homogeneous Hamiltonian Systems

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Abstract

We study the local scaling properties associated with straight line periodic orbits in homogeneous Hamiltonian systems, whose stability undergoes repeated oscillations as a function of one parameter. We give strong evidence of local scaling of the Poincaré section with exponents depending simply on the degree of homogeneity of the potential.

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It has been recognized for some time now that periodic orbits play a crucial role, whether in classical or semiclassical dynamics. Of the infinite periodic orbits the most important ones are those with the shortest periods and highest stabilities. Gutzwiller's trace formula in the semiclassical quantization of chaotic systems [1], as well as the zeta function approaches in classical and semiclassical mechanics of such systems [2], accord them the highest weights. The bifurcation properties of these orbits also assume considerable significance. Certain atomic experiments have revealed the importance of bifurcations of closed orbits even when the dynamics is chaotic [3].

Most studies of classical Hamiltonian systems have focused upon single parameter systems, upon whose variation the system smoothly undergoes a transition from regular motion to chaotic motion via stages of mixed phase spaces. It can so happen that integrability may be suddenly recovered for certain values of the parameter. However, even while the parameter variation is over a range in which the remnant tori are being destroyed and replaced by chaotic trajectories, there could be periodic orbits that are rapidly undergoing stability oscillations implying the creation of secondary tori and regular regions in the phase space. This has been known for sometime and seems to be more generic with homogeneous Hamiltonian systems [4].

Such stability oscillations occur in very simple periodic orbits and as stated above these are of importance. Homogeneous Hamiltonian systems, while rather special, allow certain simplifications that make their study useful. While in general Hamiltonian systems the orbits form one parameter families with energy being the parameter [5], in homogeneous systems varying of energy simply scales the orbits without changing the orbit structure in the phase space, that is bifurcations and related phenomena cannot occur as a function of energy, in general. Thus we resort to changing the Hamiltonian itself in the form of parameter variations.

For homogeneous Hamiltonian systems, Yoshida [4] has given an exact and simple expression for the trace of the monodromy matrix of certain

straight line periodic orbits which have in general low periods and high stability, or are among the least unstable orbits. The monodromy matrix is the linearization of the Poincaré map in the neighborhood of the periodic orbit. To fix ideas and introduce the scaling associated with these orbits, we will begin with the well studied model of the quartic oscillator given by the Hamiltonian [6]

$$H_4 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{4}\beta_1 q_1^4 + \frac{1}{4}\beta_2 q_2^4 + \frac{1}{2}\alpha q_1^2 q_2^2, \tag{1}$$

where $\beta_1,\ \beta_2,\ \alpha >\ 0$. For fixed values of the parameters $\beta_{1,2}$, as α is increased, the phase space is known to become more chaotic. Although even at $\alpha = \infty$ there are islands of stability, these are of miniscule proportions. The straight line periodic orbit, the "channel orbit", specified by the initial conditions $(p_1^0, q_1^0, 0, 0)$, is clearly one of the simplest orbits of the system and is known to play a crucial part in the semiclassics and quantum mechanics of the oscillator; for instance they scar a series of eigenfunctions which form a near WKB series even in the highly chaotic regimes [7].

The channel periodic orbits do not increasingly become unstable as α is increased, they recover stability by repeated oscillations. This implies that over whole ranges of α, however chaotic the rest of phase space may be, there are islands of stability around this periodic orbit and that in the stable regions various bifurcations give rise to new periodic orbits. For the instance of the Hamiltonian specified by Eq. 1, the Yoshida formula [4] gives

$$\mathrm{Tr}\ J(\alpha) = 2\sqrt{2}\cos(\frac{\pi}{4}\sqrt{1 + 8\frac{\alpha}{\beta_1}}), \tag{2}$$

where $J(\alpha)$ is the monodromy matrix for the *half* Poincaré map [8] of the oscillator. Thus $J(\alpha)$ is the linearized map about the channel periodic orbits specified by $(q_2 = p_2 = 0)$. Due to the symmetries of the system we are considering, namely reflection symmetries about the various axes, the half map defined as successive intersections of the trajectories with the plane $q_1 = 0$, *irrespective of whether* p_1 *is positive or negative*, is an one to one area

preserving map. We thus note that the orbit can change stability whenever $\alpha = \beta_1 m(1+2m)$ where m is any integer, as at these values Tr $J(\alpha) = \pm 2$.

For large enough α, the phase space is mostly chaotic, hence when the channel orbit is stable, its island of stability must be rapidly shrinking with α. We can compare the stable areas on the half Poincaré sections at various α, such that the central orbit stability is the same at these values, and the slope of the stability curve, dTr $(J(\alpha))/\mathrm{d}\alpha$, has the same sign. For instance Fig. 1 shows the neighborhood of the origin, corresponding to the channel orbit ($q_2 = p_2 = 0$), when its stability is just about to be lost in a pitchfork bifurcation, i.e. when Tr $J(\alpha) = 2$ and the trace is increasing. It is clear that while the islands are shrinking with the parameter, they are essentially similar and would possibly scale with α. We thus formulate our principal results, which are at present only in the form of numerical explorations, as follows.

Let

$$q_2' = f(q_2, p_2; \alpha) \quad p_2' = g(q_2, p_2; \alpha) \tag{3}$$

be the half Poincaré map. As a consequence of the reflection symmetries in the oscillator, the function f is such that $f(q_2, p_2; \alpha) = -f(-q_2, -p_2; \alpha)$, with a similar relation for g. Then the scaling of the section implies the scaling of the above functions. Let us choose two values of the parameter α and α', such that say $\alpha' > \alpha$. If α and α' are related by Tr $J(\alpha) =$ Tr $J(\alpha')$, and the stability is either increasing at both α and α', or decreasing, then

$$(\frac{\alpha'}{\alpha})^{-\gamma_1} f(q_2, p_2; \alpha) \; = \; f((\frac{\alpha'}{\alpha})^{-\gamma_1} q_2, (\frac{\alpha'}{\alpha})^{-\gamma_2} p_2; \alpha'). \tag{4}$$

Here γ_1 and γ_2 are the scaling exponents for the q_2 and p_2 directions respectively.

For the class of Hamiltonians (we call here class I), given by

$$H_{2n} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2n}(\beta_1 q_1^{2n} + \beta_2 q_2^{2n}) + \frac{\alpha}{2}(q_1^2 q_2^{2n-2} + q_2^2 q_1^{2n-2}),$$

of which the quartic oscillator used above is a special case, we conjecture the following, based on numerical evidence to be presented below:

$$\gamma_1 = \frac{2n+1}{4n} \quad \gamma_2 = \frac{2n-1}{4n}. \tag{5}$$

Thus for the Hamiltonian of Eq. 1, $\gamma_1 = 5/8$ and $\gamma_2 = 3/8$. One of the consequences of the above is that the area of the sections scale simply as α^{-1}, independently of the degree of homogeneity of the potential. A similar scaling relation is found to be true for the function $g(q_2, p_2; \alpha)$.

The validity of the above scaling relationships are restricted to a certain region around the periodic orbit, in this case around the origin of the section. The scaling is in this sense only local. We have observed that the area of stability may be safely taken as the region in which the scaling holds, although this can be a serious underestimation, as will be shown below. We will illustrate the validity of the scaling by taking one of the outer most points of the stable region of the sections when Tr $J(\alpha) = 2$ and is increasing. In this case there is one island chain consisting of eight islands, that have been earlier created, and have grown out and are near the chaotic sea (Fig. 1). We will take the distances between the origin and the central period eight orbit to verify scaling. Let the period eight orbit's intersection with the positive q_2 axes be at $d_1(\alpha)$ and with the positive p_2 axes be at $d_2(\alpha)$. Fig. 2 shows the scaling of these distances with α, i.e., $d_1(\alpha) \sim \alpha^{-\gamma_1}$, and $d_2(\alpha) \sim \alpha^{-\gamma_2}$. The lines shown are those of best fit. Their slopes are equal to -0.621 and -0.372 and are very close to those predicted by the above Eq. (5). The scaling seems to become better with increasing α, so that the first few points were neglected while calculating the slope. Increasing α leads to a deterioration of the accuracy of the numerical integrations. Hence we have used smaller step sizes of the order of 10^{-6} in a fifth order Runge-Kutta integrator for converging the exponents at these high parameter ranges.

The exponents found from the above can be used to directly verify the scaling of the half first return maps as given by the Eq. 4. In the case of the Hamiltonian of Eq. 1, Fig. 3 shows the absolute value of the difference of the

two sides of Eq. 4 for the function f, for the case when $\alpha' = 120$ and $\alpha = 66$. At these values of α the trace is 2.0 and increasing when the stability is about to be lost in a pitchfork bifurcation and there are large stable islands (Fig. 1). Comparing figure 1(a) and 1(b) with Fig. 3 indicates that the area over which the scaling remains valid is much larger than the "area of stability". A similar result is obtained in the case of the function g as well.

To emphasize this we may take the case when Tr $J(\alpha) = 2$ and decreasing, when the channel orbit is about to gain stability and create two new unstable orbits (for instance $\alpha' = 136$ and $\alpha = 78$) . In this case there is no stable island, yet the scaling of the functions is valid over a large range and the picture obtained is very close to that of Fig. 3. The scaling relation is found to be true even in the case when the central channel orbit is unstable. At this stage we note that the scaling of the functions f and g do not necessarily imply scaling of the *orbits*, as in a chaotic flow which is ergodic the phase points will explore regions in which the scaling is invalid. However, in the case when the orbits never leave the region of valid scaling, we can expect the orbits themselves to be scaling. This would happen if the central orbit were to be stable, and explains our interest in this range of parameter values, as well as the likeness in the sections of Fig. 1.

Verifying scaling of the functions is much easier than measuring distances implied in Fig. 2. Using the exponents found when Tr $J(\alpha) = 2$, we have verified using Eq. (4), the scaling laws with identical exponents *independent* of the value of the trace. Another rather efficient method of determining the exponents based on Eq. (4) is to assume a fixed initial condition with $p_2 = 0$ and searching along a range in the exponents for γ_1, as this is unaffected by the value of γ_2, and then searching for γ_2 using the γ_1 obtained from such a procedure. Identical scaling behaviour with the exponents given by Eq. (5) is observed when $\beta_1 \neq \beta_2$, i.e., when the C_{4v} symmetry of the above examples is broken into C_{2v}, and this is illustrated also in Fig. 2 for the case $\beta_1 = 0.5$ and $\beta_2 = 1.0$, when the lines of best fit have slopes equal to -.622 and -.372.

An almost identical picture is obtained when we take other class I sys-

tems. For instance we consider the Hamiltonians H_6 and H_8 whose potential energies correspond to $n = 3$ and $n = 4$ respectively, within class I. The figures analogous to Fig. 2 is shown in Fig. 4 for these oscillators. The lines are once more those of best fit, and the slopes for the sextic are -0.589 and -0.422, while for the octic potential they are -0.566 and -0.442, which are very close to the values given by Eq. (5), when we consider that by definition $\gamma_{1,2}$ are negative of the slope. Once more we find that the scaling gets to be nearly perfect for large values of α.

Potentials that contain terms which do not affect the stability of the channel orbits form different classes of Hamiltonians from those considered above. For instance, one simple set of Hamiltonians we call class II is of the form

$$H'_{2n} = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2n}(\beta_1 q_1^{2n} + \beta_2 q_2^{2n}) + \frac{1}{n}\alpha q_1^n q_2^n, \;\; n > 2.$$

The channel orbit is always marginally stable (Tr $J(\alpha) = 2$), independent of α and there is a stable region around this orbit which continuously scales with α. We found the corresponding exponents to be well predicted by the following rule:

$$\gamma_1 = \gamma_2 = \frac{1}{n-2}\,, \tag{6}$$

so that the area still scales as α^{-1} only in the case when $n = 4$. The term $q_1^n q_2^n$ ($n > 2$) is like a "gauge term" as far as the stability of the central orbit is concerned. In this class of Hamiltonians the symmetry of parity is broken when n is odd, and the potential in these cases is bounded only if $-1 < \alpha < 1$. If n is odd, the *half* Poincaré map defined earlier for class I Hamiltonians is not valid, and hence we use the usual definition of *full* Poincaré map, namely, as the successive intersections of the trajectory with the plane $q_1 = 0$ and $p_1 > 0$. For example, in case of the Hamiltonian specified by the potential $(q_1^6 + q_2^6)/6 + \alpha q_1^3 q_2^3/3$, the phase space is largely chaotic for the seemingly low values of the coupling parameters near unity. Complete chaos is however absent, not only because of the channel orbit but also due to the existence of one more stable island. In this case the exponents were found, using the

methods specified above, to be $\gamma_1 = \gamma_2 = 1$. The generalization, Eq. 6, is based on similar computations for larger values of n (up to $n = 7$).

We have briefly noted some, what we believe are new, local scaling behaviours of certain homogeneous Hamiltonian systems. The above being in the nature of preliminary numerical exploration, we cannot exhaustively comment on the classes of Hamiltonian systems with distinct scaling laws, even within the sub-class of homogeneous systems. The number of degrees of freedom we have considered in this Letter is only two and higher dimensional generalizations while interesting have not yet been explored. It is also not clear if such scaling behaviours can be observed in non-homogeneous systems with similar periodic orbits. In future work we hope to address some of these questions as well as study the semiclassical implications, if any, of such scaling.

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Figure Captions

Figure 1 Poincaré surface of section around the origin for the quartic oscillator cases a) $\alpha = 66$ and b) $\alpha = 120$, with $\beta_1 = \beta_2 = 1$.

Figure 2 Scaling of the distances for the quartic oscillator, when $\alpha' = 120$ and $\alpha = 66$, and a similar case when $\beta_1 = .5$ is also shown. The upper two and lower two lines correspond to $d_2(\alpha)$ and $d_1(\alpha)$ respectively.

Figure 3 The absolute value of the difference between the left and right hand sides of Eq. 4 for the case when Tr $J(\alpha) = 2$ and increasing, $\alpha' = 120$ and $\alpha = 66$.

Figure 4 Scaling of the distances for the case of the a) sextic and b) octic oscillators ($\beta_{1,2} = 1.0$).

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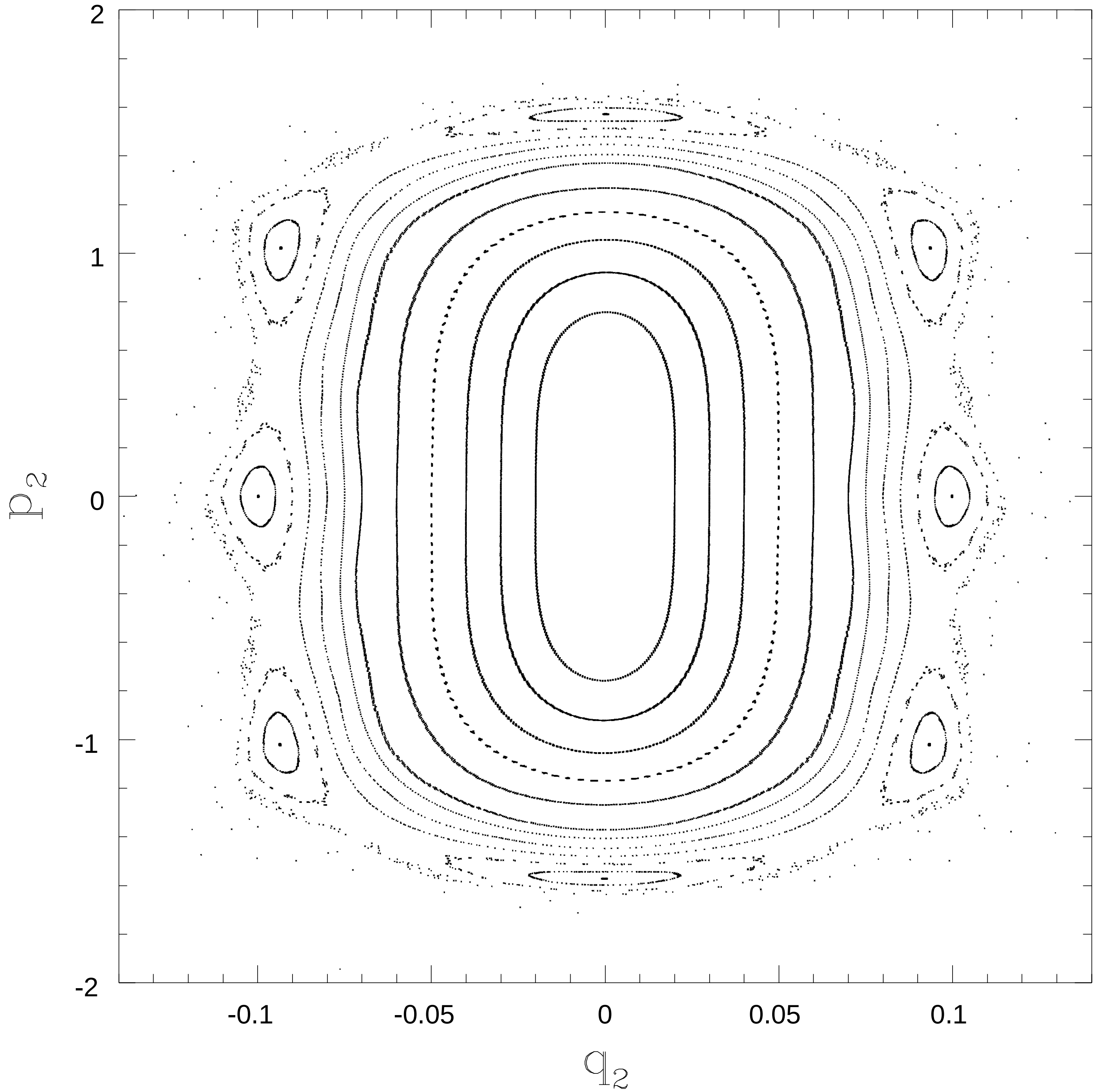
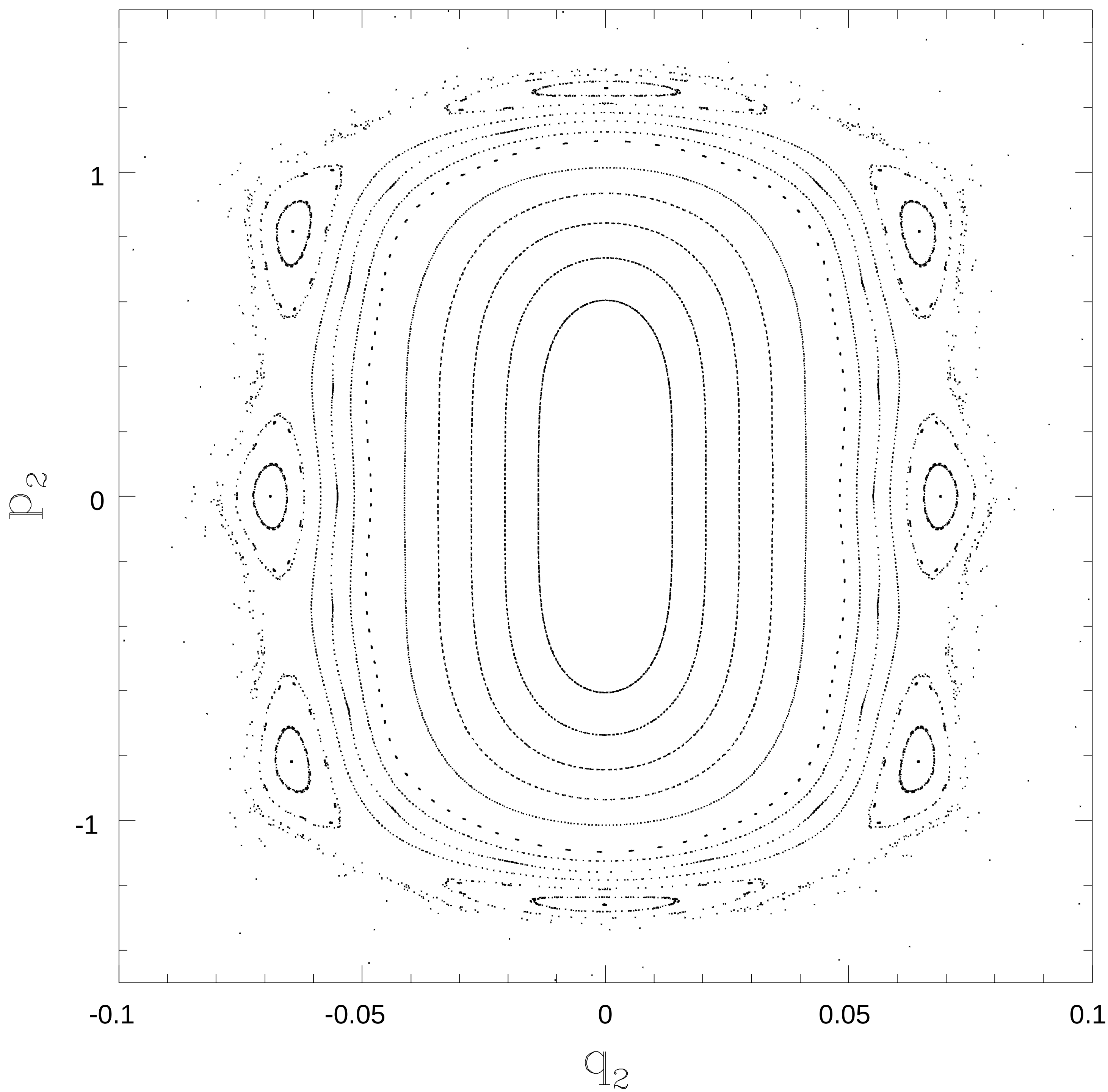
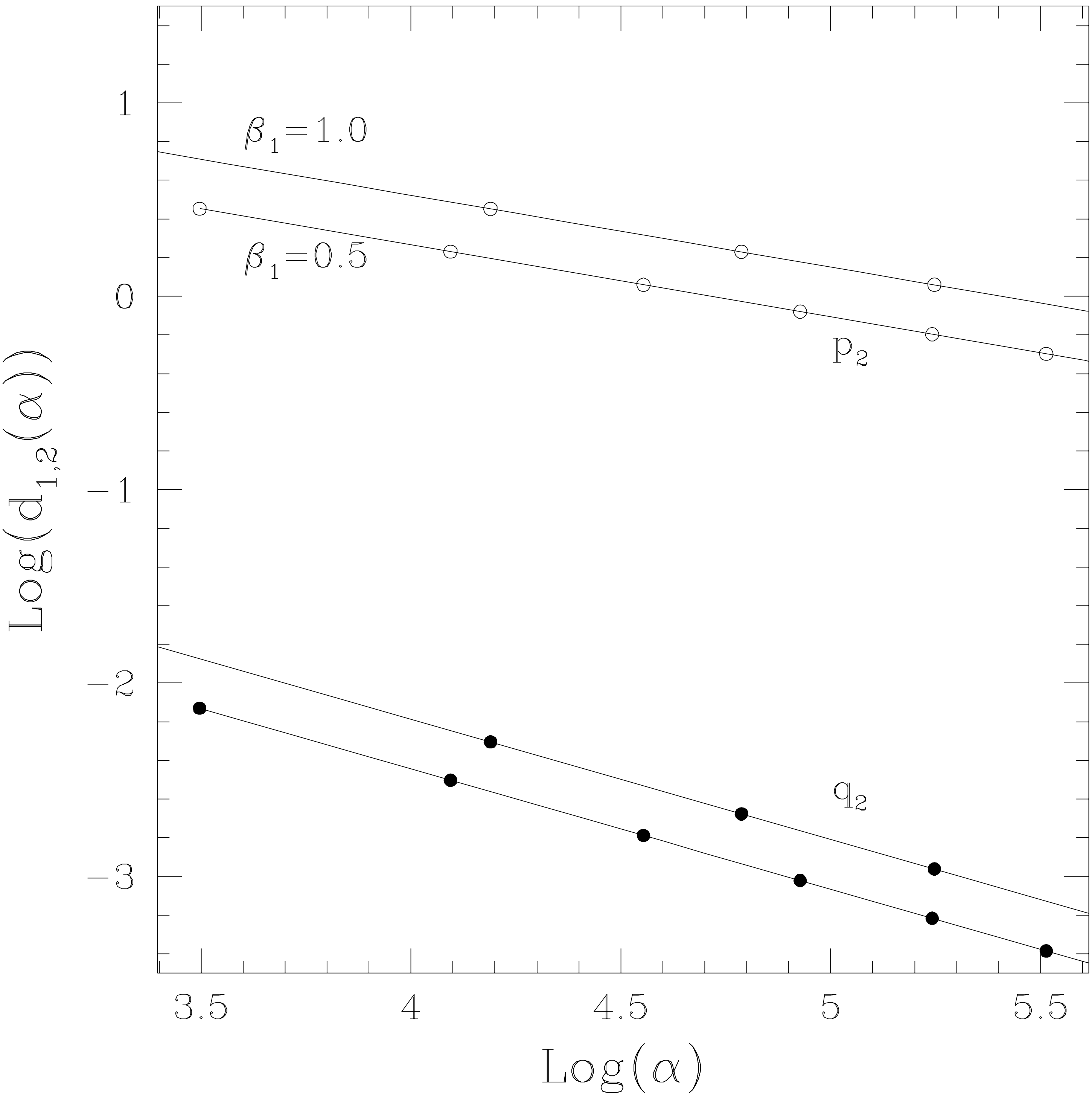


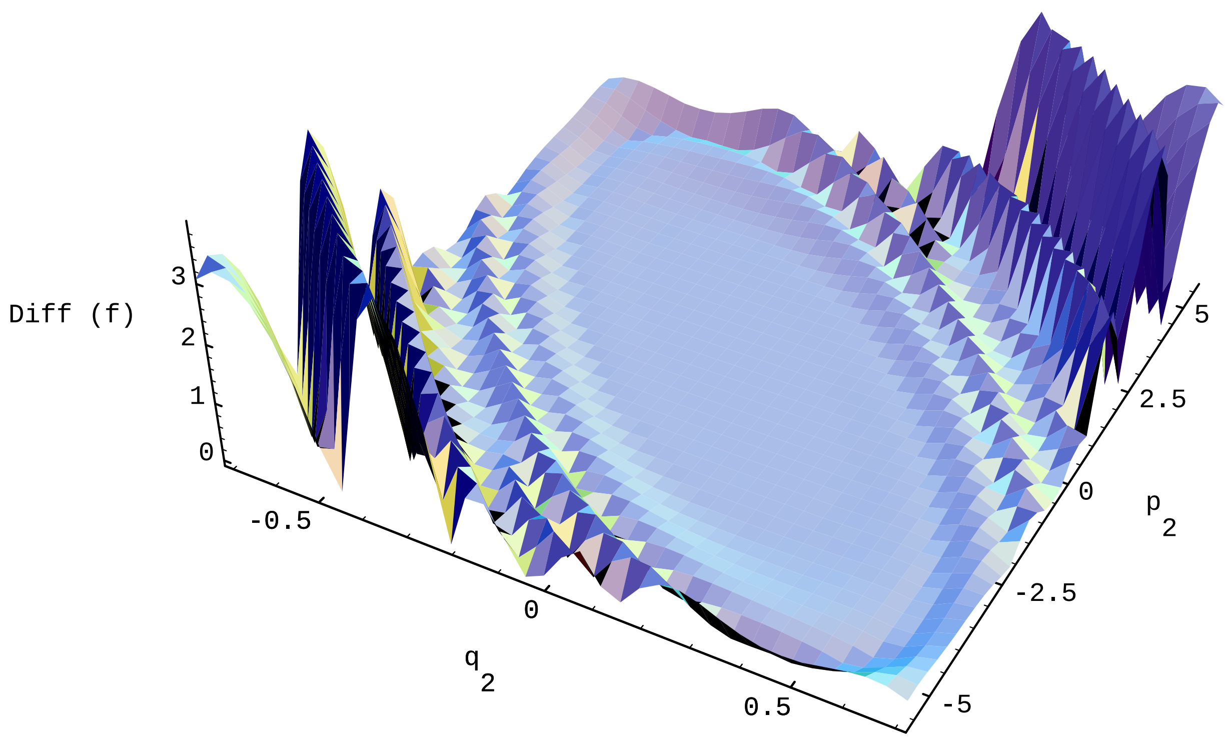
2
1
0
-1
-2
p_2
-0.1
-0.05
0
0.05
0.1
q_2



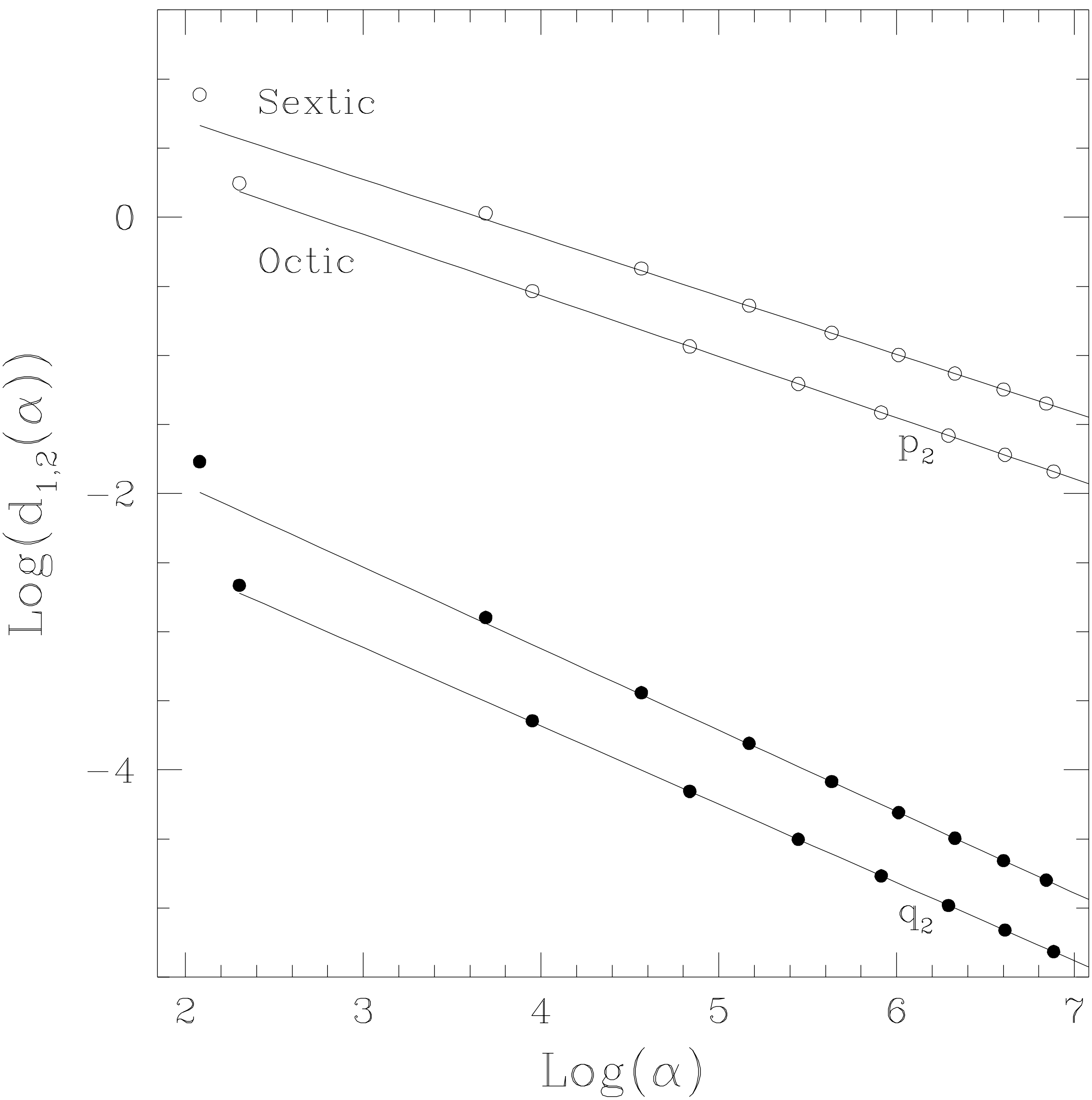
1
0
-1
p_2
-0.1
-0.05
0
0.05
0.1
q_2



$\beta_1 = 1.0$
$\beta_1 = 0.5$
p_2
q_2
$\mathrm{Log}(d_{1,2}(\alpha))$
$\mathrm{Log}(\alpha)$
1
0
−1
−2
−3
3.5
4
4.5
5
5.5



Diff (f)
3
2
1
0
-0.5
0
0.5
q
2
5
2.5
0
-2.5
-5
p
2



Sextic
Octic
p_2
q_2
0
−2
−4
2
3
4
5
6
7
$\mathrm{Log}(\alpha)$
$\mathrm{Log}(d_{1,2}(\alpha))$